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A ray determination of a class of elastodynamic diffraction coefficients

R H Tew

Division of Applied Mathematics, School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK

E-mail: richard.tew@nottingham.ac.uk

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Abstract

Ray theory is used to solve a sequence of elastic wave scattering problems, either within an isolated solid or else within two elastic solids in contact along a planar interface, with time-harmonic forcing of a Gaussian type. By appealing to the ideas behind Gaussian beam summation, we then integrate the resulting Gaussian beam solutions to produce explicit, closed-form expressions for the cylindrical fields that radiate into the elastic media as a result of Lamb-type localized boundary forcing. This approach furnishes something usually impossible from ray analyses of diffraction problems—namely the associated far-field diffraction coefficients or 'directivities'—and does so without the need to resort to delicate steepest descents and branch cut manipulations of cumbersome and complex-valued integral transform solutions. Reference is also made to applications to some problems involving non-localized interfacial forcing.

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1. Introduction

One of the most significant developments in the analysis of the scattering and diffraction of single-frequency acoustic, electromagnetic or elastic waves—governed by scalar (acoustic) or vectorial (electromagnetic and elastic) Helmholtz equations—in the short-wavelength limit is the celebrated geometrical theory of diffraction. Accounts and applications of this method, which has its foundations within ray theory, can be found in the papers by Keller [1], Keller and Lewis [2], Karal and Keller [3] and in the books by Červeneỳ [4] and Babič and Buldyrev [5] (and it should be noted that the latter contains an excellent review of the relevant Russian literature).

The principal strengths of the method lie in its applicability to a wide range of geometries in two or three dimensions, wave types (e.g., bulk and surface waves) and media (acoustic, electromagnetic or elastic, homogeneous or otherwise), and also in its highly intuitive, geometrical foundations. However, it does have deficiencies and will fail near regions of non-uniformity such as at caustics or shadow boundaries, for example. Also, although it will predict accurately the form of the phase and amplitude structure of the various wavefields present in the solution of a given scattering problem, it will not furnish explicit expressions for them in that there is generally a multiplicative diffraction coefficient within the amplitude, often a function of position or polar angles, which this ray analysis cannot fix. In order to do so, recourse must usually be made to the exact solution of an appropriate canonical diffraction problem, and a comparison made between the appropriate asymptotic limit of this solution and the aforementioned ray solution.

The former class of deficiencies can be treated using boundary or interior layer techniques and matched asymptotic expansions (see, for example, the papers by Buchal and Keller [6], Zauderer [7] and Ludwig [8]), although relatively recent developments in Gaussian beam summation have led to alternative ways of circumventing this problem. The idea is to construct the standard ray solution, and then to pose and solve an appropriate parabolic wave equation along each ray trajectory yielding a Gaussian beam profile enveloped around each of the original rays. The superposition of all of these beams—one for each constituent ray—then provides an accurate asymptotic solution to the original problem. One of the powers of the method is that it remains uniformly valid near regions where the ray solution alone would not have been, and so the separate boundary layer analyses just referred to are unnecessary. We refer to the pioneering work of Popov [9], Černeneỳ *et al* [10] and Katchalov and Popov [11], and the references cited therein, for accounts and examples of the method.

Another example worthy of mention is the novel analysis by Babich *et al* [12] of the complete high-frequency wave structure generated by a point source located close to the interface between two wave-bearing media. It had previously been noted by others (see the relevant citations listed in [12]) that in this configuration, a wavefield confined to a so-called supercritical region of one of the media is excited, and that this field apparently did not have a natural interpretation in terms of rays. However, Babich *et al* were indeed able to obtain a description of all of the fields that were generated, including the aforementioned 'non-geometrical' one, using traditional ray methods (and an application of a certain reciprocity principle). This analysis has relevance here because not only did construct a full description of a wavefield previously believed to be unobtainable by ray methods alone, but also they did so by allowing for the possibility that some of the eikonal functions that arise were complex-valued; both of these general features are also present in the calculations that we present here and we shall make further comparisons between the two approaches towards the end of this paper.

The principles behind Gaussian beam summation will play a key role in the problems that we shall consider here, which involves the determination of a certain class of diffraction coefficients in elastic wave scattering using ray-type methods alone—something which was not previously possible. To set this into context, we begin discussing the approach taken by Barbone [13] when constructing the two-dimensional acoustic fields that radiate from an impedance surface under localized line-forcing.

The specific problem considered by Barbone was to find the outgoing pressure field p satisfying the two-dimensional Helmholtz equation and impedance condition,

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + k^2 p = 0, \qquad y > 0, \tag{1}$$

$$\frac{\partial p}{\partial y} + \alpha p = \delta(x), \qquad y = 0, \tag{2}$$

respectively. Note that a time-harmonic factor $e^{-i\omega t}$ is assumed and suppressed, as we shall do throughout this paper, and that we have changed Barbone's original notation using y instead of z and α instead of β; this is to avoid notational clashes later in this paper. Of course, there will be a far-field cylindrically-spreading pressure field p_c which, in terms of plane polar coordinates (r, θ), is easily predicted to be of the form

$$p_c(r,\theta) \sim D(\theta)\frac{e^{ikr}}{(kr)^{1/2}}, \tag{3}$$

as well as a surface wave (which can be thought of as an eigensolution to (1) and the homogeneous version of (2)). The 'directivity' $D(\theta)$ of the former field and the excitation coefficient of the latter are precisely the diffraction coefficients (in this case) that cannot usually be extracted from a ray analysis alone.

Despite the existence of an exact integral transform solution of this problem, from which $D(\theta)$ can be extracted from a standard saddle-point estimate and the surface wave launching coefficient from a residue contribution, Barbone's strategy was to calculate approximate closed-form expressions for these quantities using matched asymptotic expansions and to do so in a fashion that does not rely in any way upon the existence of an exact solution (a luxury not always available in more general problems). The particular limit that was studied was $\alpha/k \to 0$, although other important limits could have been studied too.

An alternative approach to calculating explicit expressions for the far-field directivities (we shall return to the issue of surface wave coefficients at the end of the paper) of such forced scalar wave problems was proposed by the current author [14]. Essentially, the idea is to replace the boundary-value problem (e.g., (1) and (2)) by another in which the $\delta(x)$-forcing in the boundary condition is replaced by a forcing with the Gaussian profile. This 'augmented' problem is then solved using complex rays (i.e., traditional rays in which the eikonal function is allowed to be complex-valued—see the articles by Thomson [15], Kravtsov *et al* [16] and Chapman *et al* [17] for descriptions and applications of this technique) to generate a Gaussian beam solution very much in the spirit of that derived by Keller and Streifer [18]. We then consider a continuous superposition (via one straightforward integration) of these beams—each of which has physical relevance in its own right—and after doing so we are able to obtain a closed-form expression for the directivity without the need to take recourse to any exact solutions that might exist, thus avoiding having to perform delicate manipulations of cumbersome, complex-valued integrals. It is in this context that we compare the current methodology to that of the Gaussian beam summation. Although the method we use to construct our beams is different to that used by Červeneỳ *et al* [10], our overall strategy is to generate the wavefield that we are seeking by integrating over these beams; this general idea underpins the principles behind Gaussian beam summation (see equation (55) in [10]).

In [14], a sequence of scalar wave scattering problems—including one involving the coupled acoustics of two adjacent media and another concerning an extension to Barbone's problem to incorporate non-localized forcing—were studied and illustrated the robustness of the method. Our purpose here is to consider a significant extension of this methodology in terms of its application to problems in *elastic* wave scattering. The added complications here are that the field equations and boundary conditions, apart from being much more complicated in structure, permit the propagation of *two* distinct types of waves. Hence, the Gaussian beam summation procedure must be modified in order to account for the two separate types of beam generated by the boundary data in the appropriate augmented problem. We shall also see that there are other subtleties brought about by this aspect of the problem, and we shall highlight these as we meet them.

2. Far-field directivities from localized forcing

We begin by considering the case of a vacuum-backed, isotropic elastic solid undergoing localized forcing in the normal component of its boundary traction—Lamb's problem. The solid, which occupies the half-plane $y < 0$, is in a state of plane strain and its elastic displacement $\boldsymbol{u}(x, y)$ can be expressed in terms of two potential functions $\psi(x, y)$ and $\chi(x, y)$ in the form

$$\boldsymbol{u} = \nabla\psi + \nabla \times (\chi \boldsymbol{k}), \tag{4}$$

where $\boldsymbol{k}$ is a unit vector in the z-direction. The first term on the right-hand side represents the longitudinal (or 'P'-type) elastic displacement whilst the second yields the shear (or 'S'-type) component; in this configuration, the latter will be in a state of vertical polarization. The boundary value problem that we pose is to solve

$$\left(\nabla^2 + k_P^2\right)\psi = 0, \qquad y < 0, \tag{5}$$

$$\left(\nabla^2 + k_S^2\right)\chi = 0, \qquad y < 0, \tag{6}$$

$$2\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\chi}{\partial y^2} - \frac{\partial^2\chi}{\partial x^2} = 0, \qquad y = 0, \tag{7}$$

$$2\mu\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\chi}{\partial x \partial y}\right) + \lambda\nabla^2\psi = F_0\delta(x), \qquad y = 0, \tag{8}$$

with ψ and χ both outgoing as $(x^2+y^2)^{\frac{1}{2}} \to \infty$ in $y < 0$. Here, k_P and k_S are the longitudinal and shear wavenumbers, respectively, λ and μ are the usual Lamé constants, and F_0 is an $O(1)$ constant.

Equations (7) and (8) are the boundary conditions to be satisfied at the free surface of the solid. The former guarantees that the shear component of the stress τ_{xy} at the surface vanishes, so that

$$\tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \qquad \text{on} \quad y = 0,$$

where $\boldsymbol{u} = (u_x, u_y)$, whilst the latter is a model for the normal component of surface stress τ_{yy} being zero everywhere except at the origin, where we apply a delta-function forcing, leading to

$$\tau_{yy} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - 2\mu\frac{\partial u_x}{\partial x} = F_0\delta(x).$$

Using (4) to calculate u_x and u_y in terms of the potential functions ψ and χ then produces (7) and (8); full details can be found in the book by Graff [19], especially in section 6.1.1 We shall be concerned with the high-frequency solution in the limits $k_P, k_S \to \infty$ and for which $k_P/k_S = O(1)$.

In this example, we obtain the 'augmented' problem referred to in the introduction by replacing (8) by

$$2\mu\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\chi}{\partial x \partial y}\right) + \lambda\nabla^2\psi = \frac{k_P^2 F_0}{2\pi}\exp\left(\mathrm{i}k_P\beta x - \frac{k_P x^2}{2\varepsilon}\right). \tag{9}$$

Some comments regarding this are in order. First, the wavenumber k_P arises on the right-hand side of this revised boundary condition. We could just as easily have used k_S and, whilst some of the intermediate stages of the calculation that follows would then be different, the final result is the same in the either case. Second, we have introduced two new parameters into the problem, namely β and ε. Though full details are presented in [14], it turns out to be important that whilst we take ε to be a small lengthscale, we still need to insist upon the limits $k_P\varepsilon \to \infty$ and $k_S\varepsilon \to \infty$ as k_P, k_S both become large; this is to remain consistent with the ray limit that we are taking. Despite this, we can still use the smallness of ε as a basis for asymptotics elsewhere in the calculation. Given the form of this modified boundary condition (which is of a Gaussian profile), it is not unreasonable to expect it to generate beam-type elastic wavefields in the solid; we shall see that this is indeed the case and that the parameter β relates to their direction of propagation.

To construct the leading-order solution to the augmented problem, we assume a ray ansatz for each of ψ and χ in the form

$$\psi \sim A_P\,\mathrm{e}^{\mathrm{i}k_P U_P}, \qquad \chi \sim A_S\,\mathrm{e}^{\mathrm{i}k_S U_S} \tag{10}$$

in which the phases U_α and the leading-order amplitudes A_α ($\alpha = P, S$) are functions of position. Substitution into (5) and (6), respectively, and extracting terms of $O\big(k_\alpha^2\big)$ and then $O(k_\alpha)$, yields that they satisfy the eikonal equation

$$\nabla U_\alpha \cdot \nabla U_\alpha = 1 \tag{11}$$

and the transport equation

$$2\nabla A_\alpha \cdot \nabla U_\alpha + A_\alpha \nabla^2 U_\alpha = 0, \tag{12}$$

respectively, again for $\alpha = P, S$. We note here that in the standard notation of the theory of partial differential equations, (11) and (12) (together with appropriate boundary data, which we shall discuss shortly) represent the Cauchy problems for the eikonal functions U_α and, subsequently, the amplitude A_α. Although we continue to present our analysis in the spirit of traditional ray theory, we could nonetheless consider these problems in this more general setting using the methods behind solving Cauchy problems (see Sneddon [20] for details).

By equating exponents in the two ray ansätze in (10) and on the right-hand side of (9), we see that the boundary data that accompanies (11) is

$$U_P = \beta s + \frac{\mathrm{i}s^2}{2\varepsilon} \tag{13}$$

$$U_S = \frac{c_S}{c_P}\left(\beta s + \frac{\mathrm{i}s^2}{2\varepsilon}\right), \tag{14}$$

in which c_S, c_P are the shear and longitudinal wavespeeds, respectively, and we have parametrized the boundary in terms of arclength s such that $x = s$, $y = 0$. The characteristics of (11), which can be obtained by a routine application of Charpit's method (see Sneddon [20] for a general description of the method, or Tew [14] for an account relevant to this particular problem), are the rays of geometrical optics. In addition, we also require that the accumulation of all such ray contributions yields an outgoing, or else an exponentially decaying (rather than growing) solution. We refer to [14], [15] and [16] for details.

That we have two complex ray fields, driven by (13) and (14), is our first departure from the scalar problems considered in [14], and considering the 'P'-type field first, we find that the equations of the rays are

$$x = s + \tau\left(\beta + \frac{\mathrm{i}s}{\varepsilon}\right), \qquad y = -\tau\left[1 - \left(\beta + \frac{\mathrm{i}s}{\varepsilon}\right)^2\right]^{1/2}, \tag{15}$$

where the frst '$-$' sign in the y-equation guarantees propagation into $y < 0$. By eliminating τ between these equations, and introducing a modified polar angle $\Theta = -\theta\,(\pi > \Theta > 0)$, we find that the smallness of ε leads to an approximation of the launch point s of the ray passing through the point in the solid with polar coordinates (r, Θ) in the form

$$s \sim \mathrm{i}\varepsilon\,(\beta - \cos\Theta)\left(1 + \frac{\mathrm{i}\varepsilon}{r}\sin^2\Theta\right). \tag{16}$$

Further, we can show that the distance τ that the ray must travel in order to make this intersection is

$$\tau \sim r - \mathrm{i}\varepsilon\cos\Theta(\beta - \cos\Theta) \tag{17}$$

and that the phase of the contributory ray is then

$$U_p \sim r + \tfrac{1}{2}\mathrm{i}\varepsilon\,(\beta - \cos\Theta)^2. \tag{18}$$

(Note that $U_P = \tau + \beta s + \mathrm{i}s^2/2\varepsilon$, which is a consequence of Charpit's equations). The analogues of (15)–(18) for the 'S'-type rays are, respectively,

$$x = s + c\tau\left(\beta + \frac{\mathrm{i}s}{\varepsilon}\right), \qquad y = -\tau\left[1 - c^2\left(\beta + \frac{\mathrm{i}s}{\varepsilon}\right)^2\right]^{1/2}, \tag{19}$$

$$s \sim \mathrm{i}\varepsilon\left(\beta - \frac{1}{c}\cos\Theta\right)\left(1 + \frac{\mathrm{i}\varepsilon}{r}\sin^2\Theta\right), \tag{20}$$

$$\tau \sim r - \mathrm{i}\varepsilon\cos\Theta\left(\beta - \frac{1}{c}\cos\Theta\right) \tag{21}$$

and

$$U_s \sim r + \tfrac{1}{2}\mathrm{i}\varepsilon\left(\beta - \frac{1}{c}\cos\Theta\right)^2, \tag{22}$$

in which $c = c_S/c_P$.

We must now consider the amplitudes A_P, A_S and we begin by remarking that along the rays, the transport equations (12) reduce to the ordinary differential equations

$$2\frac{\mathrm{d}A_\alpha}{\mathrm{d}\tau} + A_\alpha\nabla^2 U_\alpha = 0, \qquad \alpha = P, S. \tag{23}$$

Following some algebra (to compute $\nabla^2 U_\alpha$), we find that in this case the solution is

$$A_\alpha(s,\tau) = A_\alpha(s,0)\left[\frac{q_\alpha}{\tau(p'_\alpha q_\alpha - p_\alpha q'_\alpha)}\right]^{1/2}, \qquad \alpha = P, S. \tag{24}$$

Here, p_α and q_α are the values of $\partial U_\alpha/\partial x$ and $\partial U_\alpha/\partial y$, respectively (which can be found by differentiation of (13), (14) with respect to s (to find p_α) and then using the eikonal equation (11) (to find q_α)), evaluated on the boundary, and a prime denotes differentiation with respect to s. In terms of the approximations for s and τ that we have already deduced, we may now use these in (24) to get the results

$$A_P(s,\tau) \sim A_P(s,0)\sin\Theta\,\mathrm{e}^{-\mathrm{i}\pi/4}\left(\frac{\varepsilon}{r}\right)^{1/2}, \tag{25}$$

$$A_S(s,\tau) \sim A_S(s,0)\sin\Theta\,\mathrm{e}^{-\mathrm{i}\pi/4}\left(\frac{\varepsilon}{cr}\right)^{1/2}. \tag{26}$$

Hence, we now just need to calculate the boundary amplitudes $A_\alpha(s,0)$, $\alpha = P, S$, and we do this by substituting the two ansatz's (10) into the boundary conditions (7) and (9) and then take the leading-order terms (in k_P, k_S). The upshot is

$$A_P(s,0) = \frac{F_0}{2\pi\rho c_P^2}\frac{\left(1-2q_s^2\right)}{\left[4c^2 q_S q_P p_S p_P - \left(1-2c^2 p_P^2\right)\left(1-2q_S^2\right)\right]}, \tag{27}$$

$$A_S(s,0) = \frac{F_0}{2\pi\rho c_P^2}\frac{2c^2 p_P q_P}{\left[4c^2 q_S q_P p_S p_P - \left(1-2c^2 p_P^2\right)\left(1-2q_S^2\right)\right]}, \tag{28}$$

where ρ is the density of the solid (related to the Lamé constants by $c_P^2 = (\lambda + 2\mu)/\rho$, $c_S^2 = \mu/\rho$).

At this point it is important to identify another subtlety that does not arise in the scalar cases mentioned previously. For a given observation point within the solid, there is precisely one contributory 'P'- and 'S'-type ray passing through it. However, we see from (16) and (20) that the boundary launch points of these rays are different. Hence, the values of s that are to be inserted into (27) and (28) are also correspondingly different and (using (16) and (20)) we find that in (27) we must take

$$p_P \sim \cos\Theta, \quad q_P \sim -\sin\Theta, \quad p_S \sim c\cos\Theta, \quad q_S \sim -(1-c^2\cos^2\Theta)^{1/2}, \tag{29}$$

and, for (28),

$$p_P \sim \frac{1}{c}\cos\Theta, \quad q_P \sim -\left(1-\frac{1}{c^2}\cos^2\Theta\right)^{1/2}, \quad p_S \sim \cos\Theta, \quad q_S \sim -\sin\Theta. \tag{30}$$

We have now calculated expressions for the leading-order amplitude and the phase of the augmented problem, and piecing all of the results together we find that

$$\psi \sim \frac{(2c^2\cos^2\Theta - 1)\sin\Theta}{[(1-2c^2\cos^2\Theta)^2 + 4c^3\cos^2\Theta\sin\Theta(1-c^2\cos^2\Theta)^{1/2}]}\frac{\mathrm{e}^{-\mathrm{i}\pi/4}F_0}{2\pi\rho c_P^2}\left(\frac{\varepsilon}{r}\right)^{1/2} \times \exp\left[\mathrm{i}k_P r - \frac{k_P\varepsilon}{2}(\beta-\cos\Theta)^2\right], \tag{31}$$

$$\chi \sim \frac{-2c\cos\Theta\left(1-\frac{1}{c^2}\cos^2\Theta\right)^{1/2}\sin\Theta}{\left[(2\cos^2\Theta-1)^2 + 4c\cos^2\Theta\sin\Theta\left(1-\frac{1}{c^2}\cos^2\Theta\right)^{1/2}\right]}\frac{\mathrm{e}^{-\mathrm{i}\pi/4}F_0}{2\pi\rho c_P^2}\left(\frac{\varepsilon}{cr}\right)^{1/2} \times \exp\left[\mathrm{i}k_S r - \frac{k_S\varepsilon c}{2}\left(\beta - \frac{1}{c}\cos\Theta\right)^2\right]. \tag{32}$$

We, therefore, see that we have indeed generated a pair of beams with the Gaussian profile which essentially emanate from the origin and are exponentially localized (recalling that $k_\alpha\varepsilon \gg 1, \alpha = P, S$) around the polar angles $\cos^{-1}\beta$ (for ψ, the 'P'-type field) and $\cos^{-1}(\beta c)$ (for χ, the 'S'-type field). Of course, these are exponentially small everywhere in the solid if $\beta > 1$ or $\beta c > 1$, respectively.

Our final step is to appeal to the principle of Gaussian beam summation, which in this context is interpreted to be a continuous superposition (i.e., an integration) of the solutions (31) and (32) over all real values of β. First, note that the field equations (5) and (6) and the first boundary condition (7) remain unaltered, since we can simply take the integration through the partial derivatives because of the linearity of the problem. However, integrating (9) with respect to β yields $k_P F_0 \delta(x)$ on the right-hand side and we, therefore, deduce that this superposition of our complex ray beam solution provides that for the orignal problem (5)–(8) posed, multiplied by a factor k_P. Hence, to get the leading-order far-field for this problem with $\delta(x)$-forcing, all we need to do is integrate (31) and (32) with respect to β and then divide by k_P—a straightforward procedure which leads to the expressions

$$\psi \sim \frac{(2c^2\cos^2\Theta - 1)\sin\Theta}{[(1-2c^2\cos^2\Theta)^2 + 4c^3\cos^2\Theta\sin\Theta(1-c^2\cos^2\Theta)^{1/2}]}\frac{\mathrm{e}^{-\mathrm{i}\pi/4}F_0}{\sqrt{(2\pi)}\rho c_P^2 k_P}\frac{\mathrm{e}^{\mathrm{i}k_P r}}{\sqrt{(k_P r)}} \tag{33}$$

$$\chi \sim \frac{-2\cos\Theta\left(1-\frac{1}{c^2}\cos^2\Theta\right)\sin\Theta}{\left[(2\cos^2\Theta-1)^2 + 4c\cos^2\Theta\sin\Theta\left(1-\frac{1}{c^2}\cos^2\Theta\right)^{1/2}\right]}\frac{\mathrm{e}^{-\mathrm{i}\pi/4}F_0}{\sqrt{(2\pi)}\rho c_P^2 k_p}\frac{\mathrm{e}^{\mathrm{i}k_S r}}{\sqrt{(k_S r)}}. \tag{34}$$

(We continue to use the same labels for the field variables in both the augmented and ray problems since no confusion is likely).

Inspection of (33) and (34) shows that we have generated two cylindrically-spreading elastic wavefields of the form (3), together with explicit expressions for the associated directivities; as we stated at the outset, this was achieved by one ray calculation and a trivial integration and at no point did we have to consider integrals arising from an integral transform analysis.

We can go on to use this methodology to consider problems with a similar forcing at the common interface between two elastic solids. Consider, for example, the augmented problem

$$\left(\nabla^2 + k_P^2\right)\psi^- = 0, \qquad \left(\nabla^2 + k_S^2\right)\chi^- = 0, \qquad y < 0, \tag{35}$$

$$\left(\nabla^2 + K_P^2\right)\psi^+ = 0, \qquad \left(\nabla^2 + K_S^2\right)\chi^+ = 0, \qquad y > 0, \tag{36}$$

$$\left[\!\left[2\mu\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\chi}{\partial x\partial y}\right) + \lambda\nabla^2\psi \right]\!\right] = \frac{F_0 k_P^2}{2\pi}\exp\left(\mathrm{i}\beta k_P x - \frac{k_P x^2}{2\varepsilon}\right), \qquad y = 0, \tag{37}$$

$$\left[\!\left[2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\chi}{\partial y^2} - \frac{\partial^2\chi}{\partial x^2} \right]\!\right] = 0, \qquad y = 0, \tag{38}$$

$$\left[\!\left[\frac{\partial\psi}{\partial y} - \frac{\partial\chi}{\partial x} \right]\!\right] = 0, \qquad y = 0, \tag{39}$$

$$\left[\!\left[\frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial y} \right]\!\right] = 0, \qquad y = 0. \tag{40}$$

Here, (36) represents the equations of motion (with K_P and K_S being the longitudinal and shear wavenumbers) in the 'upper' solid (the notation in $y < 0$, where the equations of motion are given in (35), being unchanged), and $[\![f]\!]$ denotes the discontinuity in the quantity f in passing from the 'lower' solid to the 'upper' solid, e.g.

$$\left[\!\left[\frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial y} \right]\!\right] = \left(\frac{\partial\psi^-}{\partial x} + \frac{\partial\chi^-}{\partial y}\right) - \left(\frac{\partial\psi^+}{\partial x} + \frac{\partial\chi^+}{\partial y}\right).$$

The boundary conditions (37)–(40) require interpretation. We now have two elastic solids in contact along a common interface $y = 0$, being subject to a surface stress distribution of Gaussian profile in the direction normal to this interface. We, therefore, require that the normal component of stress be discontinuous by an amount prescribed by this forcing (i.e. in the notation of section 2,

$$[\![\tau_{yy}]\!] = \frac{F_0 k_P^2}{2\pi}\exp\left(\mathrm{i}\beta k_P x - \frac{k_P x^2}{2\varepsilon}\right)$$

on $y = 0$), but that the shear component of stress must be continuous (i.e., $[\![\tau_{xy}]\!] = 0$), and these conditions explain (37) and (38). We also require continuity of the tangential (u_x) and normal (u_y) components of the elastic displacement and, in terms of ψ and χ, these conditions lead directly to (39) and (40), respectively.

We now pose the ray approximations

$$\psi^- \sim A_P^- \, e^{ik_P U_P^-}, \qquad \chi^- \sim A_S^- \, e^{ik_S U_S^-}, \qquad y < 0, \tag{41}$$

$$\psi^+ \sim A_P^+ \, e^{iK_P U_P^+}, \qquad \chi^+ \sim A_S^+ \, e^{iK_S U_S^+}, \qquad y > 0, \tag{42}$$

and expect four ray fields to be generated—a 'P'- and an 'S'-type in each of the two solids. Although this problem is much more complicated than the previous one, most of the mathematical machinery required to solve it is already in place (further illustrating the convenience of this approach).

Indeed, the phase calculations are essentially already done since they only depend upon the exponent in the right-hand side of (37), which was deliberately chosen to be the same as that considered in the previous problem. Given this, equations (15)–(22) inclusive carry over directly as far as the calculation of ψ^- and χ^- are concerned, and analogous expressions for s, τ, U_P^+ and U_S^+ for the fields ψ^+ and χ^+ can be deduced form (20)–(22) simply by replacing Θ by θ (the standard polar angle) and c by C_P/c_P (for ψ^+) or C_S/c_P (for χ^+). The amplitudes A_P^+ and A_S^+ also follow from (25) and (26) with the same changes. With all of this information we can now write the leading-order solution to this augmented problem as follows:

$$\psi^+ \sim A_P^+ \left(i\varepsilon \left(\beta - \frac{c_P}{C_P} \cos\theta \right), 0 \right) e^{-i\pi/4} \sin\theta \left(\frac{\varepsilon}{r} \frac{c_P}{C_P} \right)^{1/2} \times \exp\left(iK_P r - \frac{1}{2} K_P \varepsilon \frac{C_P}{c_P} \left(\beta - \frac{c_P}{C_P} \cos\theta \right)^2 \right), \tag{43}$$

$$\chi^+ \sim A_S^+ \left(i\varepsilon \left(\beta - \frac{c_P}{C_S} \cos\theta \right), 0 \right) e^{-i\pi/4} \sin\theta \left(\frac{\varepsilon}{r} \frac{c_P}{C_S} \right)^{1/2} \times \exp\left(iK_S r - \frac{1}{2} K_S \varepsilon \frac{C_S}{c_P} \left(\beta - \frac{c_P}{C_S} \cos\theta \right)^2 \right), \tag{44}$$

$$\psi^- \sim A_P^- \left(i\varepsilon \left(\beta - \cos\Theta \right), 0 \right) e^{-i\pi/4} \sin\Theta \left(\frac{\varepsilon}{r} \right)^{1/2} \exp\left(ik_P r - \frac{1}{2} k_P \varepsilon \left(\beta - \cos\Theta \right)^2 \right), \tag{45}$$

$$\chi^- \sim A_S^- \left(i\varepsilon \left(\beta - \frac{c_P}{c_S} \cos\Theta \right), 0 \right) e^{-i\pi/4} \sin\Theta \left(\frac{\varepsilon}{r} \frac{c_P}{c_S} \right)^{1/2} \times \exp\left(ik_S r - \frac{1}{2} k_S \varepsilon \frac{c_S}{c_P} \left(\beta - \frac{c_P}{c_S} \cos\Theta \right)^2 \right). \tag{46}$$

These results depend upon the boundary values of the amplitudes $A_P^+(s, 0)$, $A_S^+(s, 0)$, $A_P^-(s, 0)$ and $A_S^-(s, 0)$ and, as before, these are obtained by considering the leading-order balance when (41), (42) are substituted into (37)–(40). The system of equations that arise are

$$-\left[1 - \frac{2c_S^2}{c_P^2} \left(\frac{\partial U_P^-}{\partial x} \right)^2 \right] A_P^- + 2 \frac{\partial U_S^-}{\partial x} \frac{\partial U_S^-}{\partial y} A_S^- + \left[1 - 2 \frac{C_S^2}{C_P^2} \left(\frac{\partial U_P^+}{\partial x} \right)^2 \right] \gamma A_P^+ - 2\gamma \frac{\partial U_S^+}{\partial x} \frac{\partial U_S^+}{\partial y} A_S^+ = \frac{F_0}{2\pi\rho c_P^2} \tag{47}$$

$$-2\frac{c_S^2}{c_P^2}\frac{\partial U_P^-}{\partial x}\frac{\partial U_P^-}{\partial y}A_P^- + \left[1-2\left(\frac{\partial U_S^-}{\partial y}\right)^2\right]A_S^- + 2\frac{c_S^2}{C_P^2}\frac{\partial U_P^+}{\partial x}\frac{\partial U_S^+}{\partial y}A_P^+$$
$$-\left[1-2\left(\frac{\partial U_S^+}{\partial y}\right)^2\right]\frac{c_S^2}{C_S^2}A_S^+ = 0 \qquad (48)$$

$$\frac{c_S}{c_P}\frac{\partial U_P^-}{\partial y}A_P^- - \frac{\partial U_S^-}{\partial x}A_S^- - \frac{c_S}{C_P}\frac{\partial U_P^+}{\partial y}A_P^+ + \frac{c_S}{C_S}\frac{\partial U_S^+}{\partial y}A_S^+ = 0 \qquad (49)$$

$$\frac{c_S}{c_P}\frac{\partial U_P^-}{\partial x}A_P^- + \frac{\partial U_S^-}{\partial y}A_S^- - \frac{c_S}{C_P}\frac{\partial U_P^+}{\partial x}A_P^+ - \frac{c_S}{C_S}\frac{\partial U_S^+}{\partial y}A_S^+ = 0. \qquad (50)$$

Here, γ is the ratio of the densities ϱ/ρ (ϱ being the density in $y > 0$ and ρ remains that in $y < 0$) and the quantities $\partial U_\alpha^\pm/\partial x$, $\partial U_\alpha^\pm/\partial y$, can readily be obtained by differentiation of the boundary data for the eikonal equations.

We do not present the solutions to this system of equations partly for economy of space and partly because their precise form adds little to the general argument being developed, suffice it to say that they can be solved explicitly and inserted into (43)–(46) to give explicit expressions for the beam solutions radiating into both solids. Having done so, the analytical forms of the far-field directivities for the problem in which the right-hand side of (37) is replaced by a delta-function $\delta(x)$ follow by integration with respect to β and division by k_P. For example, that associated with the longitudinal field ψ^+ in the 'upper' medium is given by

$$D_P^+(\theta) = A_P^+\left(\mathrm{i}\varepsilon\left(\beta - \frac{c_P}{C_P}\cos\Theta\right), 0\right)\sin\Theta\,\mathrm{e}^{-\mathrm{i}\pi/4}\frac{c_P}{k_P C_P}\sqrt{2\pi}, \qquad (51)$$

with expressions for the other directivities following similarly.

3. Discussion and concluding remarks

We have presented a methodology for obtaining explicit expressions for the far-field directivities of a certain class of fully coupled elastodynamic radiation problems; and have done so using ray theory alone. The novelty lies in the technique itself and its wide-ranging versatility. We chose to work in terms of potential functions (3), but could just as easily have applied the method directly to Navier's equations of linear elasticity; exactly the same results arise. Of course, given this flexibility we could also apply the method to analogous problems in electromagnetism. Indeed, we can also consider some problems with non-localized forcing. Consider, for example, replacing the boundary condition (8) by

$$2\mu\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\chi}{\partial x\partial y}\right) + \lambda\nabla^2\psi = F_0\,\mathrm{e}^{\mathrm{i}\alpha k_P x}H(x), \qquad (52)$$

where $H(x)$ is the Heaviside function. This models the situation when half of the boundary is being oscillated at a prescribed wavenumber. Then we have that

$$\left(\frac{\partial}{\partial x} - \mathrm{i}\alpha k_P\right)\left(2\mu\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\chi}{\partial x\partial y}\right) + \lambda\nabla^2\psi\right) = F_0\delta(x) \qquad (53)$$

and we can once more apply our methods (the result being that the directivities are essentially those arising in (33) and (34) multiplied by a factor $1/(\cos\Theta - \alpha)$). Suppose next that a finite portion of length $2a$ of the interface in which $-a < x < a$ is subject to forced oscillations, such as occurs when a continuous-wave transducer is applied to the boundary. Then an appropriate model is to replace the right-hand side of (52) by $F_0\,\mathrm{e}^{\mathrm{i}\alpha k_P x}\,(H(x+a) - H(x-a))$, and an appropriate superposition of the previously obtained solutions (each with displaced argument) suffices.

Also, should we wish to consider 'higher-order' forcing (such as represented by $\delta'(x)$, for example) then differentiation (with respect to x) of the results obtained so far will yield the correct results.

Finally, we return to the work of Barbone [13] and Babich *et al* [12], noting that the elegant matching procedure developed in the former was also able to predict the launching coefficient of the subsonic surface wave that the boundary forcing excited. He was, therefore, able to solve the problem completely without ever using an integral transform (apart from optional validation purposes). The same general comment applies to the work presented in [12]. The analysis that we have presented here cannot predict directly the amplitudes of similar waves arise here (such as Rayleigh, Stoneley or head/lateral waves) and the observation that we make is that all of the information about their amplitude coefficients is embedded within the relevant directivity function. This can be seen by recalling that the directivity function is the amplitude profile of a cylindrically spreading wavefield comprising an expansion fan of rays centred on the forcing. If there are other wavefields present, then the ray solution becomes singular along those polar angles for which there is a tangency between the two families of rays (and this will be a complex angle for subsonic surface waves). A local analysis (such as that performed by Barbone) will simultaneously remove this nonuniformity and provide the as yet unknown surface wave launching coefficient. Thus, once we have determined the directivities (in the absence of any integral transforms), we can use them to determine all other wavefields in the problem, arguably in a fashion that is more intuitive and straightforward than analysing a saddle-point coalescing with a simple pole (for surface waves) or a branch point (for head waves) in a transform analysis. Work into this aspect of these problems is currently in progress.

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